Exact boundary controllability of a system of mixed order with essential spectrum

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Joint work with Farid Ammar Khodja and Arnaud Münch

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"Control of Partial Differential Equations"
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$$
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u_1'' - \Delta u_1 - \alpha \frac{\partial u_2}{\partial x} = 0 & \text{in } \Omega \times (0,T) \\
u_2'' + \alpha \frac{\partial u_1}{\partial x} + a u_2 = 0 & \text{on } \Omega \times (0,T) \\
u_1 = v 1_\Gamma & \text{on } \partial\Omega \times (0,T) \\
(u_1(\cdot,0), u_2(\cdot,0))^t = (u_1^0, u_2^0)^t = u^0 & \text{in } \Omega \\
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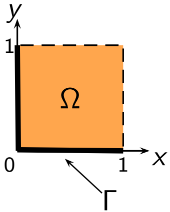
$$



The exact controllability problem

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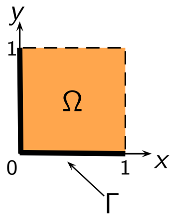

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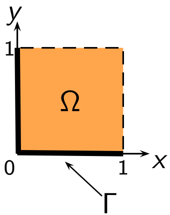

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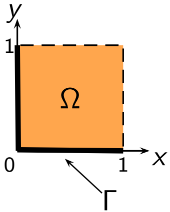

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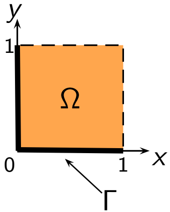

- $H = L^2(\Omega) \times L^2(\Omega)$
- $H_{1/2} = H_0^1(\Omega) \times L^2(\Omega)$
- $H_{-1/2} = H_{1/2}'$

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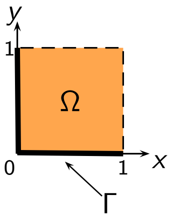

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State space: $X = H \times H_{-1/2}$

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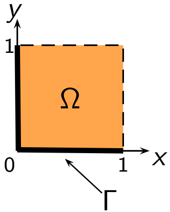

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Hypotheses: $a > \alpha^2 > 0, \ \sqrt{a - \alpha^2}/\pi \notin \mathbb{N}^*$

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T large enough, $(u^0, u^1) \in X$, $(u_T^0, u_T^1) \in X$

Exact controllability problem

$$\exists ?\, v \in L^2(\Gamma \times (0,T)) \;/\; \begin{cases} u(\cdot,T) = u_T^0 & \text{in } \Omega \\ u'(\cdot,T) = u_T^1 & \text{in } \Omega \end{cases}, \quad u = (u_1, u_2)^t$$

Adjoint system: for $(\Phi^0, \Phi^1) \in X_1 = H_{1/2} \times H$

$$\begin{cases} \varphi'' - \Delta\varphi - \alpha \frac{\partial \psi}{\partial x} = 0 \\ \psi'' + \alpha \frac{\partial \varphi}{\partial x} + a\psi = 0 \\ \varphi_{|\partial\Omega} = 0 \\ (\varphi(\cdot, 0), \psi(\cdot, 0))^t = \Phi^0 \\ (\varphi'(\cdot, 0), \psi'(\cdot, 0))^t = \Phi^1 \end{cases}$$

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Mixed order operator: $A : D(A) \subset H \to H$

$$A\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} -\Delta\left(\varphi + \alpha\Delta^{-1}\partial_x\psi\right) \\ \alpha\partial_x\varphi + a\psi \end{pmatrix}$$

$$D(A) = \left\{(\varphi, \psi)^T \in H_0^1(\Omega) \times L^2(\Omega) / \varphi + \alpha\Delta^{-1}\partial_x\psi \in H^2(\Omega)\right\}$$

$$H_{-1} = D(A)', \quad X_{-1} = H_{-1/2} \times H_{-1}$$

Well-posedness of the controlled system

$$H_{-1} = D(A)', \quad X_{-1} = H_{-1/2} \times H_{-1}$$

- **Dirichlet map:** $\mathcal{D} : D(\mathcal{D}) \subset L^2(\Gamma) \to H$ defined by $D(\mathcal{D}) = \left\{ v \in L^2(\Gamma), \mathcal{D}v \in H \right\}$ and

$$\begin{cases} \Lambda \mathcal{D} v = 0 & \text{in } \Omega \\ (\mathcal{D}v)_1 = v\, 1_\Gamma & \text{on } \partial\Omega \end{cases}, \qquad \Lambda = \begin{pmatrix} -\Delta & -\alpha\partial_x \\ \alpha\partial_x & a \end{pmatrix}$$

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$\mathcal{D}$ is an unbounded operator from $L^2(\Gamma)$ into H.

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Proposition

For every $\epsilon \in (1/2, 1]$, $\mathcal{D} \in \mathcal{L}(H^\epsilon(\Gamma), H)$.

- **Control operator:**

$$B = \begin{pmatrix} 0 \\ A\mathcal{D} \end{pmatrix} : D(B) \subset L^2(\Gamma) \to X_{-1}, \text{ with } D(B) = D(\mathcal{D})$$

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- **Maximal and dissipative operator L:**

$$L : D(L) \subset X_{-1} \to X_{-1}, \ L = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \text{ with } D(L) = X$$

- **The associated internal control problem:**
  Setting $z = (u, u')^t$,

$$(1)\begin{cases} u_1'' - \Delta u_1 - \alpha \frac{\partial u_2}{\partial x} = 0 \\ u_2'' + \alpha \frac{\partial u_1}{\partial x} + a u_2 = 0 \\ u_{1|\partial\Omega} = v 1_\Gamma \\ u(\cdot, 0) = u^0 \in H \\ u'(\cdot, 0) = u^1 \in H_{-1/2} \end{cases} \iff \begin{cases} z' = Lz + Bv \\ z(0) = (u^0, u^1)^t \in X \end{cases} \quad \text{in } X_{-1}$$

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- **Consequence of the semigroup theory:**

Proposition [Well-posedness]

For any $(u^0, u^1) \in X$ and $v \in H^1([0,T], D(\mathcal{D}))$, system (1) has a unique solution u in $H_{-1/2}$, and $u \in C([0,T],H) \cap C^1([0,T],H_{-1/2}) \cap C^2([0,T],H_{-1})$.

- **Input map:** $L_T : D(L_T) \subset L^2(\Gamma \times (0,T)) \to X$ defined by $D(L_T) = \left\{ v \in L^2(\Gamma \times (0,T)), L_T v \in X \right\}$ and

$$L_T v = \int_0^T S(T-t) B v(t) dt,$$

where $(S(t))_{t \geq 0}$ is the C^0-semigroup associated with L.

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- **Formulation of the controllability problem:** given $T > 0$, $\left(u^0, u^1\right) \in X$ and $\left(u^0_T, u^1_T\right) \in X$, to find $v \in D(L_T)$, such that $u(.,T) = u^0_T,\ u'(.,T) = u^1_T$ in Ω.

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- **Observability inequality:**

(1) exactly controllable at time $T \Longleftrightarrow \exists C(T) > 0 / \forall (\Phi^0, \Phi^1) \in D(L_T^* L)$
$$\left\| \left(\Phi^0, \Phi^1 \right) \right\|_{X_1}^2 \leq C(T) \int_0^T \int_\Gamma \left(\frac{\partial \varphi}{\partial \nu} + \alpha \psi \nu_1 \right)^2 d\sigma dt.$$

G. Geymonat, V. Valente
A noncontrollability result for systems of mixed order,
SIAM J. Control Optim. **39(3)** 661–672 (2000).

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$$\Downarrow$$

non-observability

Eigenvalues of A:

- $\lambda_{p,q}^{\pm} = \frac{1}{2}\left(\left(p^2+q^2\right)\pi^2 + a \pm \sqrt{\left((p^2+q^2)\,\pi^2 - a\right)^2 + 4\alpha^2 p^2 \pi^2}\right)$, for $p \in \mathbb{N}^*,\ q \in \mathbb{N}^*$
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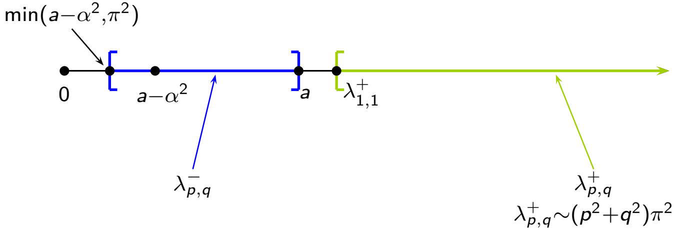
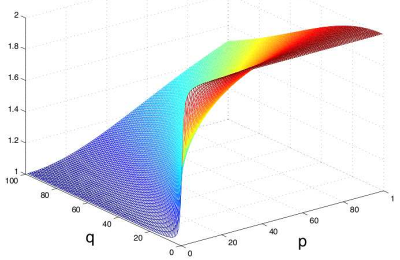
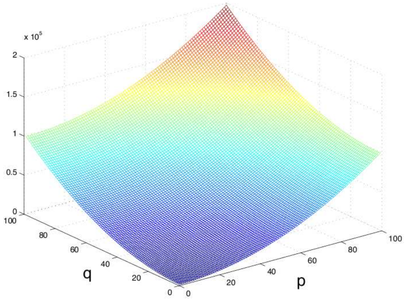


$a = 2,\ \alpha = 1$



Graph of $\left(\lambda^-_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$

$\forall (p,q)\ \ \min\left(a-\alpha^2, \pi^2\right) \le \lambda^-_{p,q} \le a$



Graph of $\left(\lambda^+_{p,q}\right)_{(p,q)\in\mathbb{N}^*\times\mathbb{N}^*}$

$\lambda^+_{p,q} \sim \left(p^2+q^2\right)\pi^2$

Definition

- Spectrum of A: $\sigma(A) = \{\lambda \in \mathbb{C} \;/\; \lambda I - A \text{ is not invertible}\}\cdot$
- Essential spectrum of A:

$$\sigma_{ess}(A) \; = \; \{\lambda \in \sigma(A) \, / \lambda \text{ is not an isolated eigenvalue with finite multiplicity}\}\cdot$$

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Proposition

$$\sigma_{ess}(A) = \left[a - \alpha^2, a\right] = \text{set of the accumulation points of } \left(\lambda^-_{p,q}\right)_{(p,q)}$$

G. Geymonat and G. Grubb
The essential spectrum of elliptic systems of mixed order
Math. Ann., **227** (1977), pp. 247–276.

Eigenvectors of A:

- $e^{\pm}_{p,q}(x,y) = \left(c^{\pm}_{p,q}\sin(p\pi x)\sin(q\pi y)\,,\, d^{\pm}_{p,q}\cos(p\pi x)\sin(q\pi y)\right)^t$
  with $c^{\pm}_{p,q} = \frac{2\left(\lambda^{\pm}_{p,q}-a\right)}{\sqrt{\left(\lambda^{\pm}_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}$ and $d^{\pm}_{p,q} = \frac{2\alpha p\pi}{\sqrt{\left(\lambda^{\pm}_{p,q}-a\right)^2+\alpha^2p^2\pi^2}}$
- $e_q(x,y) = \left(0, \sqrt{2}\sin\left(q\pi y\right)\right)^t$, associated with a

Proposition

$\left\{e^{+}_{p,q}\right\}_{p\geq 1, q\geq 1} \cup \left\{e^{-}_{p,q}\right\}_{p\geq 1, q\geq 1} \cup \left\{e_q\right\}_{q\geq 1}$ is a Hilbert basis of H.

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An observability result (1)

$X_1 = H_{1/2} \times H,$
$H^{\pm} = \overline{\mathrm{span}(\{e^{\pm}_{p,q}\}_{p,q\geq 1})}^{\,H}, \quad X_1^{\pm} = \left(H_{1/2} \cap H^{\pm}\right) \times H^{\pm}$

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The observability inequality does not hold in $D\left(L^*_T L\right) \cap X^-_1$.

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The observability inequality does not hold in $D\left(L_T^* L\right) \cap X_1^-$.

Theorem [Observability in X_1^+]

Let $\gamma = \frac{\pi\sqrt{\pi}}{4\sqrt{2\pi+|\alpha|}}$. If $a \leq 2\pi^2$ and $T > \frac{2\pi}{\gamma}\sqrt{1 + 2\frac{\left(\lambda^+_{1,1} - a + \alpha^2\right)^2}{\left(\lambda^+_{1,1} - a\right)^2}}$, then the observability inequality is satisfied for any initial data (Φ^0, Φ^1) of the adjoint system in $D\left(L_T^* L\right) \cap X_1^+$.

F. Ammar Khodja, K. M. and A. Münch
Exact boundary controllability of a system of mixed order with essential spectrum, submitted

Consider $\left(\Phi^0, \Phi^1\right) \in X_1^+ = \left(H_{1/2} \cap H^+\right) \times H^+$.

- $\Phi^0 = \sum_{(p,q)\in(\mathbb{N}^*)^2} \Phi^0_{p,q} e^+_{p,q}$ with $\sum_{(p,q)\in(\mathbb{N}^*)^2} \lambda^+_{p,q} \left(\Phi^0_{p,q}\right)^2 < +\infty$
- $\Phi^1 = \sum_{(p,q)\in(\mathbb{N}^*)^2} \Phi^1_{p,q} e^+_{p,q}$ with $\sum_{(p,q)\in(\mathbb{N}^*)^2} \left(\Phi^1_{p,q}\right)^2 < +\infty$

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Consider $\left(\Phi^0, \Phi^1\right) \in X_1^+ = \left(H_{1/2} \cap H^+\right) \times H^+.$

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Then we have proved:

Theorem [Observability in X_1^+]

Let $\gamma = \frac{\pi\sqrt{\pi}}{4\sqrt{2\pi+|\alpha|}}$. If $a \leq 2\pi^2$ and $T > \frac{2\pi}{\gamma}\sqrt{1+2\frac{\left(\lambda_{1,1}^+ - a + \alpha^2\right)^2}{\left(\lambda_{1,1}^+ - a\right)^2}}$, then there exists $C^+(T) > 0$ such that for any initial data $\left(\Phi^0, \Phi^1\right) \in D\left(L_T^* L\right) \cap X_1^+$

$$\left\|\left(\Phi^0, \Phi^1\right)\right\|_{X_1}^2 \leq C^+(T) \int_0^T \int_\Gamma \left(\frac{\partial \varphi}{\partial \nu} + \alpha \psi \nu_1\right)^2 d\sigma dt.$$

Observability in the eigenspace associated with a:

$H^a = \overline{\mathrm{span}(\{e_q\}_{q\geq 1})}^{\,H}$

For $T > \frac{\pi}{2\sqrt{a}}$, the observability inequality also holds for any initial data (Φ^0, Φ^1) in $D\left(L_T^* L\right) \cap \Big((H^a \cap H_{1/2}) \times H^a\Big)$.

- $H^N = \overline{\mathrm{span}(\{e_q\}_{q\geq 1} \cup \{e^+_{p,q}\}_{p,q\geq 1} \cup \{e^-_{p,q}\}_{1\leq p,q\leq N})}^{\,H}$
- $H^N_{-1/2} = \overline{\mathrm{span}(\{e_q\}_{q\geq 1} \cup \{e^+_{p,q}\}_{p,q\geq 1} \cup \{e^-_{p,q}\}_{1\leq p,q\leq N})}^{\,H_{-1/2}}$
- $X^N = H^N \times H^N_{-1/2}$

Proposition [Controllability]

There exists $T_0 > 0$, such that for any $N \in \mathbb{N}^*$, any $T > T_0$, any initial data $(u_0, u_1) \in X^N$ and final data $(u^0_T, u^1_T) \in X^N$ there exists $v \in D(L_T)$ such that $u(\cdot, T) = u^0_T$ and $u'(\cdot, T) = u^1_T$ in Ω.

$$\Phi^0 = \sum_{1\le p,q\le N} \alpha_{p,q} e^{\pm}_{p,q}, \quad \Phi^1 = \sum_{1\le p,q\le N} \beta_{p,q} e^{\pm}_{p,q}$$

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- **Corresponding discrete observability inequality:**

$$(\boldsymbol{A}^{\pm}\boldsymbol{\Phi}^{\pm}, \boldsymbol{\Phi}^{\pm})_{\mathbb{R}^{2N^2}} \le C^{\pm}_N(T)(\boldsymbol{B}^{\pm}\boldsymbol{\Phi}^{\pm}, \boldsymbol{\Phi}^{\pm})_{\mathbb{R}^{2N^2}}$$

- **Discrete observability constant:**

$$C_N^{\pm}(T) = \max\{\lambda > 0,\ \boldsymbol{A}^{\pm}\boldsymbol{\Phi} = \lambda \boldsymbol{B}^{\pm}\boldsymbol{\Phi},\ \boldsymbol{\Phi} \in \mathbb{R}^{2N^2} \setminus \{0\}\}$$

- Evolution of $C_N^{\pm}(T)$ with respect to N for $(a, \alpha, T) = (4, 1, 3)$:

	N=5	N=10	N=20	N=40	N=80
$C_N^+(T)$	5.01×10^{-1}	5.43×10^{-1}	5.71×10^{-1}	5.95×10^{-1}	6.02×10^{-1}
$C_N^-(T)$	2.42×10^{1}	4.41×10^{2}	3.24×10^{3}	8.6×10^{4}	1.01×10^{6}

Thank you for your attention!